

Algebraic models for social networks

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1. Glossary

Social network. A *social network* comprises a set of relationships among members of a set $N = \{1, 2, \dots, n\}$ of actors. It can be represented by an $n \times n$ binary array X recording the presence or absence of a social relationship, or *tie*, between each pair (or ordered pair) of members of $N = \{1, 2, \dots, n\}$. If there is a relationship from actor k to actor l , we write $X(k, l) = 1$; otherwise, $X(k, l) = 0$. If the relationship is a property of a pair of actors, the network is *nondirected*; if it is a property of an ordered pair, the network is termed *directed*. The directed network X may also be regarded as a *binary relation* R_X on the set N with $(k, l) \in R_X$ if and only if $X(k, l) = 1$; equivalently, it may be construed as a *directed graph* with *node set* N and *arc set* R_X , with an *arc* from node k to node l if and only if $(k, l) \in R_X$.

Affiliation network. An *affiliation network* is an $n \times g$ binary array X recording the membership of each of a set N of actors in a prescribed set G of *groups*, with $X(k, l) = 1$ if actor k is a member of group l , and $X(k, l) = 0$ otherwise.

Multiple network. A *multiple network* is a collection of networks for each of a set of r relations. We let $X_m(k, l) = 1$ if the tie from k to l corresponding to the relation of *type* m is present; and $X_m(k, l) = 0$ if the tie is absent. Nodes k and l are joined by a *labeled walk* with *label* $Y_1 Y_2 \dots Y_j$ if there is a sequence of nodes $k = k_0, k_1, \dots, k_j = l$, for which $Y_h(k_{h-1}, k_h) = 1$ for $h = 1, 2, \dots, j$.

Local network. The *(1-)neighbourhood* of a subset P of actors in a network is defined to be the set $P \cup \{l \in N: X_m(k, l) = 1 \text{ for some } k \in P \text{ and some relation } m\}$. The *q-neighbourhood* of P is then defined recursively as the 1-neighbourhood of the $(q-1)$ -neighbourhood of P . The *q-local network* of the subset P of N is the network restricted to its q -neighbourhood.

Algebra. A (*partially ordered*) algebra is a triple $[S, F, \leq]$, where S is a nonempty set of elements (usually assumed to be finite), F is a specified set of operations, f_α , each mapping a power $S^{n(\alpha)}$ of S into S , for some non-negative finite integer $n(\alpha)$, and \leq is a partial order on S . Each operation f_α is assumed to be *isotone* in each of its variables: that is, if $x_i \leq y_i$ ($x_i, y_i \in S$; $i = 1, 2, \dots, n(\alpha)$), then $f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}) \leq f_\alpha(y_1, y_2, \dots, y_{n(\alpha)})$. A *family of algebras* is a collection of algebras each having the same set F of operations and satisfying a specified set of postulates. Two algebras belonging to the same family are termed *similar*.

Partial algebra. A *partial algebra* is a triple $[S, F, \leq]$, where S is a nonempty set of elements (usually assumed to be finite), F is a specified set of *partial* operations, f_α , each mapping some subset $T^{(\alpha)}$ of $S^{n(\alpha)}$ into S , for some non-negative finite integer $n(\alpha)$, and \leq is a partial order on S . Each partial operation f_α is assumed to be *isotone* in each of its variables: that is, if $x_i \leq y_i$ ($x_i, y_i \in S$; $i = 1, 2, \dots, n(\alpha)$), then $f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}) \leq f_\alpha(y_1, y_2, \dots, y_{n(\alpha)})$ provided that both $(x_1, x_2, \dots, x_{n(\alpha)})$, $(y_1, y_2, \dots, y_{n(\alpha)}) \in T^{(\alpha)}$. A *family of partial algebras* is a collection of algebras each having the same set F of partial operations defined on the same subsets $T^{(\alpha)}$ of the power sets $S^{n(\alpha)}$.

Semigroup. A (*partially ordered*) semigroup is an algebra $[S, F, \leq]$ in which F comprises a single binary operation f satisfying the *associativity* condition:

$$f(f(x, y), z) = f(x, f(y, z)).$$

Lattice. A *lattice* L is an algebra $[S, F, \leq]$ in which F comprises two associative and commutative binary operations, \wedge and \vee (termed *meet* and *join*, respectively) satisfying the identities:

$$x \wedge x = x, x \vee x = x$$

and

$$x \wedge (x \vee y) = x \vee (x \wedge y) = x.$$

The operations are isotone, so that $x \leq y$ is equivalent to the pair of conditions:

$$x \wedge y = x \text{ and } x \vee y = y,$$

and the operations of meet and join may be interpreted as the greatest lower bound and least upper bound, respectively. A lattice L is *distributive* if the identity

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

holds. A lattice L is *modular* if, whenever $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Role algebra. A *role algebra* is an algebra $[S, F, \leq]$ in which F comprises a single binary composition operation satisfying the condition:

$$s \leq t \text{ in } S \text{ implies } su \leq tu \text{ in } S, \text{ for any } u \in W.$$

Homomorphism. An (*isotone*) *homomorphism* from an algebra $A = [S, F, \leq]$ onto a similar algebra $B = [T, F, \leq]$ is a mapping $\varphi: S \rightarrow T$ such that,

(i) for all $f_\alpha \in F$ and $x_i \in S$,

$$\varphi(f_\alpha(x_1, x_2, \dots, x_{n(\alpha)})) = f_\alpha(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{n(\alpha)})); \text{ and}$$

(ii) $x \leq y$ in S implies $\varphi(x) \leq \varphi(y)$ in T .

The algebra B is termed a (*homomorphic*) *image* of A , and we write $B = \varphi(A)$. Each homomorphism φ from $A = [S, F, \leq]$ onto $B = [T, F, \leq]$ has a corresponding binary relation

π on S (termed here a π -relation) in which $(x,y) \in \pi$ if and only if $\varphi(y) \leq \varphi(x)$. The equivalence relation σ_π defined by $(x,y) \in \sigma_\pi$ if and only if $(x,y) \in \pi$ and $(y,x) \in \pi$ is termed a *congruence relation*.

Homomorphism lattice. The *homomorphism lattice* $L(A)$ of the algebra $A = [S,F,\leq]$ is the collection of all homomorphisms of A partially ordered by the relation: $\varphi_1 \leq \varphi_2$ if, for all $x, y \in S$, $\varphi_2(x) \leq \varphi_2(y)$ implies $\varphi_1(x) \leq \varphi_1(y)$. The *lattice* $L_\pi(A)$ of π -relations on A , dual to $L(A)$, has the partial ordering: $\pi_1 \leq \pi_2$ if $(x,y) \in \pi_1$ implies $(x,y) \in \pi_2$, for any $x, y \in S$.

2. Definition of the subject and its importance

Algebraic models have been proposed to represent structure in social networks. They are usually constructed from a set of operations and relations defined on network constituents, such as paths or walks in the network, or vectors of ties directed to or from individual network members. The algebra represents the relationships among these constituents, for example, relations of ordering among all possible walks in a multiple network, or relations of overlap and ordering among profiles of group membership. To date, they have been used to represent kinship structures [9,14,16], role structures and stability in multiple networks [2,8,10,13], states of diffusion processes in networks [33], informal hierarchy [24], connectivity structures [20] and the structure of membership in affiliation networks [23, 39]. In several cases, partial algebraic representations have also been proposed [37,40,41].

Algebraic models have usually been proposed as exact structural representations and, in many cases, therefore, their application to network data has been coupled with some form of prior data aggregation such as blockmodeling [13,50]; see also the entry on *Positional Analyses and Blockmodeling* in this volume. More rarely, algebraic construction and data aggregation have been combined into a single step [2,35].

3. Introduction

Algebraic models for relational structures such as networks have their foundations in anthropology and were developed for general network structures by Lorrain, White and colleagues [7,29,13]. These general network representations based on path structures in networks emerged from more specific models for path structures in kinship systems, including permutation group models for Australian aboriginal kinship systems [9]. Boyd extended this approach in an important way when he proposed that the structural evolution of kinship systems could be analysed in terms of homomorphic mappings [14]; applications to network semigroups were described later [13,38].

Lattice representations for relational structures have a long history in mathematics [1] and applications to relational data in many domains have been developed extensively by Wille and collaborators in formal concept analysis [4,25,51]. They have also been applied to affiliation network structures [23,46,48].

The form of algebraic structure regarded as appropriate in a particular context is a matter for substantive consideration. The large majority of forms that have been

proposed to date have depended upon the importance of paths or walks in the network, the symmetric or asymmetric nature of network ties, and the patterns of overlap among tie partners. A focus on one or more of these features has resulted in algebraic groups, semigroups, partially ordered semigroups, local role algebras and lattices as representations of different aspects of structure in multiple social networks and affiliation networks [2,8,9,13,21,23,32] and it is on these forms that we focus here.

4. Algebras from networks

A language for discriminating among relations. We begin with a finite set $\Sigma = \{x_1, x_2, \dots, x_r\}$ of r relational terms. We term Σ a relational *alphabet* and call each term x_i in Σ a *letter* of the alphabet. The letters x_1, x_2, \dots denote relational terms such as “friendship”, “co-worker”, “confidant”, and so on. If $r = 1$, the set Σ simply identifies a single network relation.

In order to construct algebraic representations that are sensitive to social structural form, claims in the social network literature can be used to introduce a *free word algebra* on the alphabet Σ . The algebra is intended to provide a language for describing more complex relational terms that can be constructed from the primitive set Σ and so to provide the means for discriminating among social structural forms. The claims about appropriate ways of constructing new terms from the primitive set may be drawn from a number of sources, but one set of such claims is helpfully laid out by White [49]. White argues that there are three universal ways in which types of tie (i.e., relational terms) are discriminated. The first is in terms of patterns of overlap; the second in terms of asymmetry of relation; and the third in terms of the institutionalisation of indirect ties. The overlap of two relational ties refers to their joint occurrence and can arguably be described using an *intersection* operation, \cap . The asymmetry of a tie can be described by the relationship between the tie and its *converse*, that is the coding of whether, for two individuals k and l , there is a tie from k to l and from l to k . The tie is *symmetric* if the tie from k to l occurs only in conjunction with the tie from l to k , and the tie is *strictly asymmetric* if a tie from k to l occurs only in the absence of a tie from l to k . We use the expression x' to denote the converse of the relational term x . Indirect ties may be described by a *composition*, or *concatenation* operation, denoted \circ . If x_1 refers to the term “friendship” and x_2 to “coworker”, then the concatenation $x_1 \circ x_2$ may be used to refer to a relation that holds between a person and his or her friend’s co-workers.

We denote by F the set of operations from which our descriptive language is to be built. Unless otherwise specified, we assume here that F comprises the three operations just described; that is $F = \{\cap, ', \circ\}$. Each operation f_α in F possesses an *arity*, $n(\alpha)$, that determines the form of new terms created from Σ by f_α : in particular, f_α leads to expressions of the form $f_\alpha(y_1, y_2, \dots, y_{n(\alpha)})$; $y_m \in \Sigma$, $m = 1, 2, \dots, n(\alpha)$. The operations \cap and \circ have arity 2, that is, they are *binary* and lead to expressions that can be written as $y_1 \cap y_2$, and $y_1 \circ y_2$, respectively, whereas the converse operation $'$ is *unary* (that is, it has an arity of 1) and leads to an expression that can be denoted y_1' (where $y_1, y_2 \in \Sigma$).

We describe a recursive construction of a free word algebra $W = [\Sigma, F]$ from the alphabet Σ and the set F of operations [1]. We call each letter in Σ an F -polynomial of rank 0. For any positive integer ρ , we define an F -polynomial of rank ρ recursively as an expression, termed a *word*, of the form $f_\alpha(u_1, \dots, u_{n(\alpha)})$, where each u_j is an F -polynomial of rank $\leq \rho-1$ (and at least one u_j is an F -polynomial of rank $\rho-1$). The (*free*) *word algebra* $W = [\Sigma, F]$ comprises all distinct F -polynomials of finite rank (where $f_\alpha(u_1, \dots, u_{n(\alpha)})$ and $f_\beta(v_1, \dots, v_{n(\beta)})$ are distinct unless $\alpha = \beta$ and $u_k = v_k$ for all $k = 1, \dots, n(\alpha)$).

It is sometimes convenient to restrict attention to the collection of words of rank no greater than some fixed integer ρ . So we also define $W_\rho = [\Sigma, F]_\rho$ as the subset of the free word algebra $W = [\Sigma, F]$ comprising words of rank $\leq \rho$.

For example, suppose that F comprises the operations $\{\cap, ', \circ\}$ and that $\Sigma = \{x, y\}$. Then the elements of rank 0 in the word algebra $W = [\Sigma, F]$ are $\{x, y\}$; the elements of rank 1 are $\{x \cap x, x \cap y, y \cap x, y \cap y, x', y', x^\circ x, x^\circ y, y^\circ x, y^\circ y\}$; elements of rank 3 include $x \cap x \cap x$, $x \cap (x^\circ x)$, $(x^\circ x) \cap x$, $x \cap x'$, $x' \cap x$, etc; and so on. In this case, the free word algebra $W = [\Sigma, F]$ comprises all composite relational terms that can be constructed from the primitive terms x and y using the intersection, converse and concatenation operations.

The free word algebra simply provides a language that permits certain forms of discrimination among types of tie to be encoded. It makes no structural assumptions whatever, although we might anticipate from the intended interpretation of the terms in F that certain terms in W will not need to be discriminated. For instance, we might suppose that: the intersection operation can be assumed to be *idempotent* ($x \cap x = x$, for all $x \in W$), *commutative* ($x \cap y = y \cap x$, for all $x, y \in W$) and *associative* ($(x \cap y) \cap z = x \cap (y \cap z)$, for all $x, y, z \in W$); that the converse operation satisfies $x'' = x$, for all $x \in W$; and that the concatenation operation is also associative. Rather than impose such postulates at this abstract level, though, we move directly to the evaluation of the terms in W for an empirical realisation of a (multiple) network.

Algebras for multiple networks. The language provided by the free algebra $W = [\Sigma, F]$ can be used to determine all possible discriminations among ties that our structural claims suggest as appropriate. To describe the structural forms to which observed network ties give rise, each of the derived relations for an observed multiple network can be computed. There is a straightforward correspondence between the words u in $W = [\Sigma, F]$ expressed in terms of letters in Σ and operations in F , on the one hand, and Boolean matrix manipulations of the binary constituents in the multiple network, on the other. The correspondence is achieved by replacing:

- the relational term x_m by the array X_m ;
- the intersection operation \cap by Boolean intersection of arrays (i.e., element-wise Boolean multiplication, denoted \bullet); and
- the composition operation \circ by Boolean matrix multiplication.

Boolean intersection, transposition and composition are defined for $n \times n$ binary arrays Y_1 and Y_2 by:

- $Y_1 \bullet Y_2(k, l) = Y_1(k, l) Y_2(k, l)$;
- $Y_1'(k, l) = Y_1(l, k)$; and

$$\bullet \quad Y_1 Y_2(k,l) = Y_1(k,1)Y_2(1,l) + Y_1(k,2)Y_2(2,l) + \dots + Y_1(k,n)Y_2(n,l).$$

The term $u \in W$ then corresponds to an $n \times n$ Boolean array, denoted X_u , with $X_u(k,l) = 1$ if and only if there is a relational tie described by the word u from k to l . We also define a partial order on the set of relations $\{X_u: u \in W\}$ by:

$$X_u \leq X_v \text{ if and only if } X_u(k,l) \leq X_v(k,l) \text{ for all } k, l \in N.$$

Suppose that we evaluate X_u for all $u \in W$. We define a generalised version of the *Axiom of Quality* by setting $u \leq v$ in W whenever $X_u \leq X_v$ [13].

Defining an induced partition θ on W by $(u,v) \in \theta$ if and only if $u \leq v$ and $v \leq u$ and letting $S = W/\theta$ be the set of associated classes with the induced partial order:

$$s \leq t \text{ in } S \text{ if } u \in s, v \in t \text{ and } u \leq v \text{ in } W,$$

we can then define a general (partially ordered) algebraic structure on S .

A (*partially ordered*) algebra is a triple $[S,F,\leq]$, where S is a nonempty set of elements (usually assumed to be finite), F is a specified set of operations, f_α , each mapping a power $S^{n(\alpha)}$ of S into S , for some non-negative finite integer $n(\alpha)$, and \leq is a partial order on S . Each operation f_α is assumed to be *isotone* in each of its variables: that is, if $x_i \leq y_i$ ($x_i, y_i \in S$; $i = 1, 2, \dots, n(\alpha)$), then $f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}) \leq f_\alpha(y_1, y_2, \dots, y_{n(\alpha)})$. A *family of algebras* is a collection of algebras each having the same set F of operations and satisfying a specified set of postulates. Two algebras belonging to the same family are termed *similar*.

For $F = \{\circ\}$, the algebra $[S,F,\leq]$ is a *partially ordered semigroup* [8].

The semigroup of a multiple network. For multiple networks, a composite relation constructed as the concatenation of observed relations records the presence of a particular form of *labelled walk*. Specifically, actor k is connected to actor l by a walk with label u if and only if $X_u(k,l) = 1$. If $X_u \leq X_v$, two actors who are joined by a walk with label u are also necessarily joined by a walk with label v .

An algorithm for constructing the semigroup of a multiple network $[S,\{\circ\},\leq]$ is as follows:

- (S1) Let $W_1 = \{X_1, X_2, \dots, X_r\}$ and set $i = 1$.
- (S2) Construct the networks with edge sets WZ for each $W \in W_i$ and each $Z \in W_1$. Place WZ in W_{i+1} if $WZ \neq Y$, for any $Y \in W_j$, $j = 1, 2, \dots, i+1$; otherwise, $WZ = Y$ for some $Y \in W_j$ ($j = 1, 2, \dots, i+1$) is an *equation* in the semigroup S .
- (S3) If $W_{i+1} = \emptyset$, stop and compute the partial order among elements in $W_1 \cup W_2 \cup \dots$; otherwise, set i to $i+1$ and return to step S2.

The equations $WZ = Y$ may be recorded in a *right multiplication table* in which Y appears as the table entry corresponding to the row labelled W and the column labelled Z . The partial ordering on S is constructed from the collection of networks in W_1, W_2, \dots .

Lattices from networks. Lattice representations for networks arise in two important ways. In the first, elements of the lattices are elements of $S = W/\theta$ for $F \supseteq \{\cap, \circ\}$. The meet of two elements is their intersection, and their join is the least relation in S which is greater than or equal to both of them. Indeed, the algebra $[S, F, \leq]$ is both a lattice and a partially ordered semigroup in this case.

In the second type of lattice construction, the elements correspond to rows or columns of an affiliation network. Since the affiliation array provides an explicit representation of the co-constitution of groups by actors and actors by groups, the lattice structures to which it gives rise can be used to analyse the duality of actors and groups [17,22,23,34,35,39,48].

A lattice denoted $L(X)$ is generated by the rows of a matrix X under the intersection operation; it is termed the *Galois* or *concept lattice* of X [4]. In $L(X)$, the meet of two elements is equal to their intersection, while their join is the minimal element in the lattice which is greater than or equal to both of them. A lattice can also be constructed from the rows of the matrix X under the union operation. In this lattice, termed the *Zareckii lattice* by Boyd, the join of two vectors is equal to their union, and their meet is the maximal vector which is less than or equal to both of them [2].

To construct the Galois lattice, one simply needs to compute all distinct intersections among rows of X , add the vector $[1 \ 1 \ \dots \ 1]$ and compute the partial order among the resulting set of row vectors.

Role algebras from networks. Role algebras have been proposed as local network analogues of the semigroup representation of relational structure in an entire social network. The representation has been presented in several slightly different forms but each is derived from an original formulation proposed by Mandel [18,32,53,54].

Denote by k^*X_u the vector $[X_u(k,1), X_u(k,2), \dots, X_u(k,n)]$ for $u \in W$. The vector indicates whether there is a relationship of type u from actor k to each other actor in the network. As before, we can envisage evaluating k^*X_u for all $u \in W$. We then define a k -centred version of the *Axiom of Quality* by setting

$$u \leq v \text{ in } W \text{ whenever } k^*X_u \leq k^*X_v$$

and an induced partition θ on W by $(u,v) \in \pi$ if and only if $u \leq v$ and $v \leq u$. Also as before, we let $S = W/\theta$ denote the set of associated classes with the induced partial order:

$$s \leq t \text{ in } S \text{ if } u \in s, v \in t \text{ and } u \leq v \text{ in } W.$$

The algebra $[S, F, \leq]$ is termed a *role algebra*.

The local role algebra for actor k may be constructed as follows.

(RA1) Let $W_1 = \{X_1, X_2, \dots, X_r\}$ and set $i = 1$.

(RA2) Construct the vectors k^*WZ for each $W \in W_i$ and each $Z \in W_1$. Place WZ in W_{i+1} if $k^*WZ \neq k^*Y$, for any $Y \in W_j$, $j = 1, 2, \dots, i+1$; otherwise, $k^*WZ = k^*Y$ for some $Y \in W_j$ ($j = 1, 2, \dots, i+1$) and $WZ = Y$ is an *equation* in the role algebra.

(RA3) If $W_{i+1} = \emptyset$, stop and compute the partial order among elements in

$W_1 \cup W_2 \cup \dots$; otherwise, set i to $i+1$ and return to step RA2.

Partial algebras. To achieve algebraic closure, it may be necessary to compute relations corresponding to words of high rank in W and, in the case of some representations at least, it can be argued that some restriction on these repeated operations should be imposed. Mandel, for instance, made such an argument for the composition operation in social networks; he claimed that short network paths are likely to be more salient to the members of the network than longer ones [32]. These considerations can be formalised with the introduction of partial algebraic structures, whose elements are subject to rank restrictions on some or all of the operations used to construct the algebra.

Let $W_\rho = [\Sigma, F]_\rho$ be the subset of the free word algebra $W = [\Sigma, F]$ comprising words of rank $\leq \rho$. Define a partial order on the elements of W_ρ by the generalised Axiom of Quality:
 $u \leq v$ in W_ρ whenever $X_u \leq X_v$.

The relation \leq is clearly both *reflexive* ($u \leq u$ for all u) and *transitive* ($u \leq v$ and $v \leq w$ implies $u \leq w$ for any u, v, w), that is, a *quasi-order*. Further, each operation f_α is *isotone* in each of its variables whenever the result of the operation is contained in W_ρ ; that is, if $x_i \leq y_i$ ($x_i, y_i \in W_\rho$; $i = 1, 2, \dots, n(\alpha)$), then $f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}) \leq f_\alpha(y_1, y_2, \dots, y_{n(\alpha)})$, provided that both $f_\alpha(x_1, x_2, \dots, x_{n(\alpha)})$, $f_\alpha(y_1, y_2, \dots, y_{n(\alpha)}) \in W_\rho$. Thus it follows that the partial ordering \leq on W_ρ gives rise to a *partial algebra* in the following sense.

A *partial algebra* is a triple $[S, F, \leq]$, where S is a nonempty set of elements (usually assumed to be finite), F is a specified set of *partial* operations, f_α , each mapping some subset $T^{(\alpha)}$ of $S^{n(\alpha)}$ into S , for some non-negative finite integer $n(\alpha)$, and \leq is a partial order on S . Each partial operation f_α is assumed to be *isotone* in each of its variables: that is, if $x_i \leq y_i$ ($x_i, y_i \in S$; $i = 1, 2, \dots, n(\alpha)$), then $f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}) \leq f_\alpha(y_1, y_2, \dots, y_{n(\alpha)})$ provided that both $(x_1, x_2, \dots, x_{n(\alpha)})$, $(y_1, y_2, \dots, y_{n(\alpha)}) \in T^{(\alpha)}$. A *family of partial algebras* is a collection of algebras each having the same set F of partial operations defined on the same subsets $T^{(\alpha)}$ of the power sets $S^{n(\alpha)}$.

Partial semigroup algebras may be constructed from labelled walks in a multiple network of some maximum length ρ [41]. For role algebras, the partial role algebra for a node k derived from W_ρ is the role algebra associated with the ρ -local neighbourhood of k [40].

5. Algebraic structure

In some cases, it is possible to analyse the structural properties of algebraic representations of networks.

The semigroup of a (single) network under composition. We first consider the semigroup for a single network array X . We say that there is a *walk* from actor k to actor l if $X(k, l) = 1$. There is a walk of *length* p from k to l if there is a sequence of actors $k = k_0, k_1, \dots, k_p = l$ such that $X(k_{i-1}, k_i) = 1$, for $i = 1, 2, \dots, p$. It is readily seen that $X^p(k, l) = 1$ if and only if there is a walk of length p from k to l . The sequence $X, X^2,$

..., generated by the composition operation is therefore a sequence recording the presence of walks of different lengths between all pairs of network actors. The semigroup structure of this sequence therefore describes relations of equality and ordering among walks in the network.

The sequence can be described with the help of the following definition. Let p be the least integer such that

$$X^p = X^q \text{ for some } q > p.$$

Also let $q = p + d$ ($d \geq 1$) be the least integer satisfying this relation. The integers p and d are termed the *index* and *period* of the network, respectively. It can readily be seen that the sequence has the form

$$X, \dots, X^{p-1}, X^p, \dots, X^{p+d-1}, X^p, \dots, X^{p+d-1}, \dots$$

The set $\{X, X^2, \dots, X^{p+d-1}\}$ defines a semigroup S of which $G = \{X^p, \dots, X^{p+d-1}\}$ is a group (defined below).

It is natural to ask what is the relationship between properties of a network and its index and period.

For a strongly connected network X (that is, a network in which there is a walk of some length from each actor to each other actor), the period d is the greatest common denominator of all integers m such that m is the length of a cycle in X (that is, a walk with the same initial and final node). If X has more than one strong component, its period is the least common multiple of the periods of those strong components [6].

A network X with index 1 and period 1 is termed *idempotent*; for a network of index p and period 1, the network X^p is idempotent. It may be shown that every idempotent network is a pseudo-order, defined as follows [44]. Let Z be a quasi-order on N and let $e_Z = Z \cap Z'$. An actor $k \in N$ is termed *Z-strict* if there is no distinct actor $l \neq k$ such that $Z(k, l) = 1$. The subset H of N is *Z-permissible* if each of its actors is *Z-strict* and there are no pairs $k, l \in N$ such that k covers l or l covers k (recall that k covers l if $Z(k, j) = 1$ and $Z(j, l) = 1$ implies $j = k$ or $j = l$). Define the relation Z_H by

$$Z_H(k, l) = 1 \text{ iff } k = l \text{ and } k \in H.$$

Then a network X is a *pseudo-order* if

$$X = Z Z_H$$

for some quasi-order Z and *Z-permissible* subset H of N .

Structure of multiple network semigroups. In some cases, networks define semigroups with certain structural features.

An element s in a semigroup S possesses a (generalised) *inverse* if there exists some $t \in S$ such that $sts = s$ and $tst = t$. The element t is termed the inverse of s . If every $s \in S$ possesses an inverse, then S is termed an *inverse semigroup*. If, in addition, S possesses an *identity* element, that is, an element e such that for any $s \in S$, there exists some $t \in S$ such that $ts = e$, then S is a *group*. The element t is termed a (*group-*)*inverse* of s . An element s in a semigroup S is *regular* if there exists some $t \in S$ such that $sts = s$. The semigroup S is *regular* if each of its

elements is regular.

It is clear from these definitions that any group is also an inverse semigroup and that any inverse semigroup is also regular. The results below describe some of the conditions under which multiple networks give rise to regular semigroups, inverse semigroups and groups.

We first describe the conditions under which S is a group [30,43]. Let X be a network relation having a group-inverse Y , that is, a relation Y such that $XY' = I$ where I is the identity relation (for which $I(k,l) = 1$ if $k = l$; $I(k,l) = 0$ otherwise). Then it follows that (i) $XX' = I = X'X$ and $Y = X$ (that is, X' is a two-sided group-inverse for X and is unique); and (ii) X is a permutation relation (that is a relation in which, for each node k , there is exactly one node l for which $X(k,l) = 1$ and exactly one node j for which $X(j,k) = 1$). Hence if each relation X_i of a multiple network is a permutation relation and X_i' is included in S for each i , then the semigroup S is a group.

An important class of social relations which are constructed as permutation relations are "marriage class systems". These systems were proposed by White as models for kinship relations in certain Australian aboriginal tribes [9]. In such systems, persons are uniquely and permanently assigned to a *clan* and clans are related to one another by marriage and descent rules which can be represented as permutation relations. The marriage relation specifies the clan containing the wives of the male members of each clan, while the descent relation identifies the clan containing the children of male members of each clan. It follows that marriage class systems give rise to groups that represent the kinship structures of the societies concerned.

Another structure arising in the description of kinship systems is the inverse semigroup [16]. One can verify whether a network semigroup is an inverse semigroup by determining whether each of its elements has an inverse in S . The following is a characterisation of conditions under which a relation possesses an inverse [6].

A network relation X has a *Thierrin-Vagner inverse* if there exists a relation Y satisfying $XYX = X$ and $YXY = Y$. Clearly, X has a Thierrin-Vagner inverse if and only if X possesses an inverse in S . If X is a regular element of S , so that $XYX = X$ for some Y in S , then YXY is a Thierrin-Vagner inverse for X . Thus determining whether X is regular will reveal whether X has an inverse. The regularity of X can be ascertained as follows.

The network relation Y is a *subinverse* of X if $XYX \leq X$. The set of subinverses of X is closed under union (that is, if Y and Z are subinverses of X then so is the relation $Y \cup Z$); hence there is a largest subinverse X^* for any relation X . The largest subinverse X^* of a relation X can be calculated from X according to $X^* = (X^T X^c X^T)^c$ where X^c denotes the complement of X . It therefore follows that a network X is regular if and only if $X \leq X(X^T X^c X^T)^c X$, and that the largest Thierrin-Vagner inverse of a regular relation X is $X^* X X^*$ [6,45].

The algebraic structure of local role algebras. Less is known of the general algebraic properties of local role algebras. It is easily seen, however, that any orderings among relations present in the semigroup of a multiple network are also present in the local role algebra of each actor in the network. As a result, a number of the algebraic properties of local role algebras are inherited from the "parent" network semigroup. For example, if $s \leq t$ in the partially ordered semigroup S constructed from the composition operation, then $s \leq t$ in the local role algebra of every node in the network. Indeed, the converse is readily seen to also hold, so that $s \leq t$ in S if and only if $s \leq t$ in the local role algebra of every node in the network.

6. Algebraic analysis

In order to describe an algebraic representation, general procedures for analysing finite algebraic representations into simpler, independent components have been developed [4,8,38,40]. These procedures have been applied to the decomposition of lattice representations [4], partially ordered semigroups and local role algebras [8,36,37]. The procedures extract maximally independent components of a given algebraic representation and lead to a detailed structural analysis.

The account of these procedures for decomposing a finite algebra into simpler components is based on definitions adapted from [1,3]. Clearly, any decomposition procedure depends on the synthesis rules by which the components are assumed to be combined to produce the algebra. Two important synthesis rules are the direct and subdirect product.

Let $A_1 = [S_1, F, \leq]$, $A_2 = [S_2, F, \leq]$, ..., $A_q = [S_q, F, \leq]$ be a collection of similar algebras. The *direct product* $A_1 \times A_2 \times \dots \times A_q$ of A_1, A_2, \dots, A_q is the algebra comprising the set $S_1 \times S_2 \times \dots \times S_q$ and the operations $f_\alpha \in F$ given by

$$f_\alpha([x_1, x_2, \dots, x_q], \dots, [z_1, z_2, \dots, z_q]) = [f_\alpha(x_1, \dots, z_1), \dots, f_\alpha(x_q, \dots, z_q)].$$

The partial order for $A_1 \times A_2 \times \dots \times A_q$ is given by

$$[x_1, x_2, \dots, x_q] \leq [z_1, z_2, \dots, z_q]$$

if and only if

$$x_1 \leq z_1, x_2 \leq z_2, \dots, \text{ and } x_q \leq z_q.$$

An algebra $B = [T, F, \leq]$ is a *subalgebra* of $A = [S, F, \leq]$ if T is a subset of S (possibly empty) which is closed under the operations of F (that is, $f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}) \in T$, for any $x_1, x_2, \dots, x_{n(\alpha)} \in T$, and any $f_\alpha \in F$); in addition, $x \leq y$ in B if and only if $x \leq y$ in A , for any $x, y \in T$. A subalgebra $C = [S, F, \leq]$ of a direct product $A_1 \times A_2 \times \dots \times A_q$ of similar algebras $A_i = [S_i, F, \leq]$ ($i = 1, 2, \dots, q$) is a *subdirect product* of A_1, A_2, \dots, A_q if for each $x_i \in S_i$, there exists an element $c \in S$ having x_i as its component in A_i .

In the case of both the direct and subdirect product of algebras, the operations in the composite algebra are performed as the conjunction of their independent operation in the component algebras. The difference between the two constructions lies in the set on which these operations are defined. For the direct product, the appropriate set is the full Cartesian product of elements from each component algebra; in the subdirect product case, the appropriate set is simply a subset of the full Cartesian product with

the additional condition that each element in each component algebra appears in *some* element of the subset.

Some well-known theorems in universal algebra establish that the existence and nature of direct and subdirect representations of an algebra are determined by the lattice of homomorphisms of the algebra, or, equivalently, by a lattice of relations on the algebra associated with its homomorphisms.

An (*isotone*) *homomorphism* from an algebra $A = [S, F, \leq]$ onto a similar algebra $B = [T, F, \leq]$ is a mapping $\varphi: S \rightarrow T$ such that,

(i) for all $f_\alpha \in F$ and $x_i \in S$,

$$\varphi(f_\alpha(x_1, x_2, \dots, x_{n(\alpha)})) = f_\alpha(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{n(\alpha)})); \text{ and}$$

(ii) $x \leq y$ in S implies $\varphi(x) \leq \varphi(y)$ in T .

The algebra B is termed a (*homomorphic*) *image* of A , and we write $B = \varphi(A)$.

Further, each homomorphism φ from $A = [S, F, \leq]$ onto $B = [T, F, \leq]$ has a corresponding binary relation π on S (termed here a π -*relation*) in which $(x, y) \in \pi$ if and only if $\varphi(y) \leq \varphi(x)$; it may readily be established that π is transitive and reflexive (and hence a quasi-order) and that the equivalence relation σ_π defined by $(x, y) \in \sigma_\pi$ if and only if $(x, y) \in \pi$ and $(y, x) \in \pi$ has the substitution property, namely:

$$(x_1, y_1), (x_2, y_2), \dots, (x_{n(\alpha)}, y_{n(\alpha)}) \in \sigma_\pi$$

implies

$$(f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}), f_\alpha(y_1, y_2, \dots, y_{n(\alpha)})) \in \sigma_\pi.$$

The relation σ_π is termed a *congruence relation*. The collection of π -relations may be partially ordered by: $\pi_1 \leq \pi_2$ if $(x, y) \in \pi_1$ implies $(x, y) \in \pi_2$, for any $x, y \in S$. Under this partial ordering, the relations π form a lattice $L_\pi(A)$, the *lattice of π -relations* on A .

The conditions under which an algebra A may be represented as a direct or subdirect product of similar algebras A_1, A_2, \dots, A_r (that is, possesses *direct* or *subdirect representations*) can be described as follows.

Let $X = \{\pi_1, \pi_2, \dots, \pi_q\}$ be a set of π relations in $L_\pi(A)$ for which

$$\pi_1 \wedge \pi_2 \wedge \dots \wedge \pi_q = \pi_{\min},$$

where π_{\min} is the minimal element of $L_\pi(A)$. Then A may be represented as a subdirect product of the algebras $\varphi_1(A), \varphi_2(A), \dots, \varphi_q(A)$, where $\varphi_i(A)$ is the image of A under the homomorphism φ_i corresponding to the relation π_i ($i = 1, 2, \dots, q$). If, in addition, for all i ,

(a) $\pi_1 \wedge \pi_2 \wedge \dots \wedge \pi_{i-1}$ and π_i are *permutable* (i.e., $(\pi_1 \wedge \pi_2 \wedge \dots \wedge \pi_{i-1}) \pi_i = \pi_i (\pi_1 \wedge \pi_2 \wedge \dots \wedge \pi_{i-1})$); and

(b) $(\pi_1 \wedge \pi_2 \wedge \dots \wedge \pi_{i-1}) \vee \pi_i = \pi_{\max}$, where π_{\max} is the maximal element of $L_\pi(A)$,

then A may be represented as the direct product of the algebras $\varphi_1(A), \varphi_2(A), \dots, \varphi_q(A)$ [1].

An algebra which possesses no nontrivial representation as the direct product of similar algebras is termed *directly irreducible*, while an algebra which cannot be expressed as a nontrivial subdirect product of similar algebras is *subdirectly irreducible*. Pattison and Bartlett advocated the use of subdirect representations, but sought to restrict attention to a small set of such representations rather than to the (possibly large) class of all

subdirect representations of the algebra [38]; see also [15]. Their restriction entailed (a) identifying all *irredundant* subdirect representations of an algebra, and then (b) defining a partial ordering on the irredundant representations and selecting only minimal members of the resulting partially ordered set. The resulting set of minimal, irredundant subdirect representations were termed the factorisations of the algebra.

More formally, an element x of a lattice L is *meet-irreducible* if $x = x_1 \wedge x_2$ implies $x = x_1$ or $x = x_2$. A subset of lattice elements $X = \{x_1, x_2, \dots, x_r\}$ is *irredundant* if

- a) each x_i is meet-irreducible; and
- b) $x_1 \wedge \dots \wedge x_{i-1} \wedge x_{i+1} \wedge \dots \wedge x_q \neq x_{\min}$, for each $i = 1, 2, \dots, q$; where x_{\min} is the minimal element of the lattice L .

The collection of all irredundant subsets of a lattice L may be partially ordered by the relation:

$X \leq Y$ iff, for each $j = 1, 2, \dots, p$, there exists some i ($i = 1, 2, \dots, q$) such that $y_j \leq x_i$ in L , where $X = \{x_1, x_2, \dots, x_q\}$ and $Y = \{y_1, y_2, \dots, y_p\}$.

A *factorisation* of an algebra A is then the subdirect representation corresponding to any minimal, irredundant subset of elements of $L_{\pi}(A)$ whose meet is the minimal relation π_{\min} .

An algorithm for factorisation. A general algorithm for constructing the set of all factorisations of an algebra has also been developed [38]. The algorithm operates on a subset of relations in the lattice $L_{\pi}(A)$ and conducts a reasonably efficient search for subsets of π relations which satisfy the conditions of the factorisation definition. In particular, the algorithm constructs the set of factorisations of an algebra A from the atoms of its lattice $L_{\pi}(A)$ of π -relations and their maximal meet-complements.

An *atom* of a lattice L is an element that covers x_{\min} . A *meet-complement* of an element $x \in L$ is an element x^* such that $x^* > x_{\min}$ and $x \wedge x^* = x_{\min}$. A meet-complement x^* of x is *maximal* if x has no other meet-complement z such that $z > x^*$.

Let $Z = \{z_1, z_2, \dots, z_a\}$ be the set of atoms of $L_{\pi}(A)$. Define the collection of sets of the form $\{z_1^*, z_2^*, \dots, z_a^*\}$ where z_i^* is a maximal meet-complement of z_i . Then it can be shown that any factorisation of A is a subset of such a set [8]. This result is the basis for the algorithm for finding factorisations. Under certain circumstances, we can write down an explicit expression for π -relations in the factorisation of A .

Let A be an algebra with π -relation lattice $L_{\pi}(A)$. Let π_{ab} be the least π -relation in which $a \geq b$ for elements $a, b \in A$; we term π_{ab} the π -relation generated by the ordering $a \geq b$.

Let z be an atom of $L_{\pi}(A)$; define the relation $\pi(z) = \{(a, b) : \pi_{ab} \wedge z = \pi_{\min}\}$. If z has a unique maximal meet-complement, then $\pi(z)$ is a π -relation and is a unique maximal meet-complement of z . Conversely, if $\pi(z)$ is a π -relation, then it is the unique maximal meet-complement of z . Further, if $Z = \{z_1, z_2, \dots, z_a\}$ is the set of atoms of $L_{\pi}(A)$ and if, for each $z_i \in Z$, z_i has a unique maximal meet-complement $\pi(z_i)$, then the factorisation of A is unique and corresponds to the π -relations $\{\pi(z_1), \pi(z_2), \dots, \pi(z_a)\}$ [8]. In other words, if

the factorisation is unique, it can be identified immediately from the maximal meet-complements of the atoms of $L_\pi(A)$.

Under some circumstances, we can make more general claims about the uniqueness of factorisations. Not surprisingly, the structure of the lattice $L_\pi(A)$ plays a major part. It is known, for instance, that if the lattice $L_\pi(A)$ is distributive, then A possesses a unique irredundant subdirect representation, and hence a unique factorisation. In particular, since the lattice of π -relations for a lattice is necessarily distributive, any finite lattice has a unique factorisation. If the lattice $L_\pi(A)$ is modular, then any irredundant subdirect representation of A has the same number of components [1].

Factorisation for partial algebras. Analogous decompositions can be described for partial algebras. An (*isotone*) *homomorphism* from a partial algebra $A = [S, F, \leq]$ onto a similar partial algebra $B = [T, F, \leq]$ is a mapping $\varphi: S \rightarrow T$ such that,

(i) for all $f_\alpha \in F$ and $x_i \in S$,

$$\varphi(f_\alpha(x_1, x_2, \dots, x_{n(\alpha)})) = f_\alpha(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{n(\alpha)}))$$

whenever the operations on both sides of the expression are defined; and

(ii) $x \leq y$ in S implies $\varphi(x) \leq \varphi(y)$ in T .

The algebra B is termed a (*homomorphic*) *image* of A , and we write $B = \varphi(A)$.

Further, each homomorphism φ from $A = [S, F, \leq]$ onto $B = [T, F, \leq]$ has a corresponding binary relation π on S (termed here a π -*relation*) in which $(x, y) \in \pi$ if and only if $\varphi(y) \leq \varphi(x)$; it may readily be established that π is transitive and reflexive (so that π is a quasi-order). Further, the equivalence relation σ_π defined by $(x, y) \in \sigma_\pi$ if and only if $(x, y) \in \pi$ and $(y, x) \in \pi$ has the substitution property, namely:

$$(x_1, y_1), (x_2, y_2), \dots, (x_{n(\alpha)}, y_{n(\alpha)}) \in \sigma_\pi$$

implies

$$(f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}), f_\alpha(y_1, y_2, \dots, y_{n(\alpha)})) \in \sigma_\pi$$

whenever the latter term is defined. The relation σ_π is often termed a *congruence relation*. The collection of all π -relations form a lattice $L_\pi(A)$, the *lattice of π -relations* on A , under the partial ordering: $\pi_1 \leq \pi_2$ if $(x, y) \in \pi_1$ implies $(x, y) \in \pi_2$, for any $x, y \in S$.

In the case of partial algebras, it is convenient to express the algebra as an intersection of the π -relations corresponding to maximally independent homomorphic images of the algebra. Recall that the minimal element π_{\min} of the lattice $L_\pi(A)$ corresponds to the algebra A itself. Thus, any expression of π_{\min} in the form

$$\pi_1 \wedge \pi_2 \wedge \dots \wedge \pi_q = \pi_{\min}$$

expresses A as the intersection of π -relations corresponding to homomorphic images of A . Any such expression that involves π -relations $\{\pi_i; i = 1, 2, \dots, q\}$ satisfying the conditions of the factorisation definition yields an expression in terms of a set of simple and maximally independent images of the algebra. Further, the same algorithm for constructing factorisations of an algebra that was described earlier may be used to find expressions of this type.

7. Algebraic and network mappings

The semigroup of a multiple network records the orderings and equations among different types of labelled paths in the network. A natural question to arise is how

properties of a multiple network constrain the relational structure of its semigroup. We therefore review some general conditions under which the semigroup of one network is a homomorphic image of the semigroup of another.

We define two multiple networks to possess the same structure if their semigroups are isomorphic, that is, if the network relations are in one-to-one correspondence and they have identical right multiplication tables and partial orders.

One strategy for investigating whether networks have similar structures is to determine the conditions under which network homomorphisms induce homomorphisms of the network semigroup [12]. A *network homomorphism* from a multiple network $\{X_1, X_2, \dots, X_r\}$ on actor set N to a multiple network $\{Y_1, Y_2, \dots, Y_r\}$ on actor set M is a mapping ψ from N onto M such that: (a) $X_m(k, l) = 1$ implies $Y_m(\psi(k), \psi(l)) = 1$, for any k, l, m ; and (b) $Y_m(i, j) = 1$ for some i, j implies that $X_m(k, l) = 1$, for some k, l such that $\psi(k) = i$ and $\psi(l) = j$. The network on M is termed the *image* of the network on N under the mapping ψ .

The mapping ψ satisfies the *structural equivalence* condition if for any m , and for any $k, l \in N$, $\psi(k) = \psi(l)$ if and only if:

- $X_m(k, j) = 1$ iff $X_m(l, j) = 1$ for any $j \in N$; and
- $X_m(j, k) = 1$ iff $X_m(j, l) = 1$ for any $j \in N$.

Lorrain and White observed that if two multiple networks are related by a network homomorphism satisfying the structural equivalence condition, then their semigroups are isomorphic and we may argue that they possess the *same* relational structure [29]. More generally, we can ask whether a homomorphism between two networks induces a homomorphism between their semigroups. It is readily seen that a homomorphism is not always guaranteed.

Network homomorphisms induced by certain blockmodels lead to semigroup homomorphisms (see the entry on *Positional Analysis and Blockmodeling* in this volume). More general conditions under which a semigroup homomorphism is guaranteed were established by Kim and Roush [28].

A network homomorphism ψ satisfies Kim and Roush's *condition G_i* if the following holds for any pair of equivalence classes ρ_1 and ρ_2 on N induced by the mapping ψ (so that $k, l \in \rho_h$ for some h iff $\psi(k) = \psi(l)$). Let the number of elements in ρ_1 and ρ_2 be n_1 and n_2 , respectively. Let D be any subset of ρ_1 of i elements or, if $i > n_1$, let $D = \rho_1$. Then, for any $m = 1, 2, \dots, r$, the set $\{l: l \in \rho_2 \text{ and } X_m(k, l) = 1 \text{ for some } k \in D\}$ has at least $\min(i, n_2)$ elements. The condition G_1 is also known as the *outdegree condition* and G_n is also termed the *indegree condition*. A network homomorphism that satisfies both G_1 and G_n is termed *regular*.

Kim and Roush demonstrated that if ψ is a network homomorphism from one multiple network onto another that satisfies the condition G_i , then there is a homomorphism mapping the semigroup of the first onto the semigroup of the second.

A more general condition combines the condition G_i with what Pattison termed the *central representatives condition* [36]. Let ψ be a network homomorphism. Then ψ satisfies the condition G_{im} if, for each class ρ of elements of N induced by ψ ,

- there exists a *central subset* C of ρ such that for any X_m
 - $X_m(k,l) = 1$ for some $k \in \rho$ implies $X_m(k^*,l) = 1$ for some $k^* \in C$, and
 - $X_m(l,k) = 1$ for some $k \in \rho$ implies $X_m(l,k^*) = 1$ for some $k^* \in C$; and
- if C^* denotes the union of central subsets C , then the central subsets C satisfy Kim and Roush's condition G_i on the network defined on C^* .

If each central subset C comprises a single actor, then the condition is equivalent to Pattison's central representatives condition, while if each central subset C comprises the whole of the equivalence class on N induced by ψ , it is equivalent to the condition G_i . Kim and Roush showed that if one network can be mapped onto another by a network homomorphism satisfying the condition G_{im} , then there is a homomorphism from the semigroup of the first to the semigroup of the second [28]. The condition G_{im} is the most general condition known that guarantees the existence of such a homomorphism.

Comparing network semigroups. Two network semigroups S_1 and S_2 are strictly comparable only if $S_1 \leq S_2$ or $S_2 \leq S_1$. Loosely speaking, though, they are "similar" if they share many homomorphic images. Boorman and White proposed that the the largest shared homomorphic image (the so-called *joint homomorphism*) is a useful construction for comparing network semigroups [13]. Bonacich and McConaghy, on the other hand, argued that the smallest semigroup containing each of S_1 and S_2 (the *common structure semigroup*) was a more appropriate representative of common semigroup structure [11,31]. The resolution of the problem of finding a representative of common structure depends on how a semigroup is viewed [8]. If a semigroup is seen as a list of the homomorphic images that it admits, then the joint homomorphism corresponds to the list of shared features, while if it seen as a collection of semigroup equations and orderings, then the common structure semigroup serves as a representation of shared structure. The common structure semigroup of two network semigroups is the semigroup of the disjoint union of the underlying networks.

Both the joint homomorphic image and common structure semigroup constructions have been shown to be useful in particular applications, the former in identifying common reductions of the networks giving rise to identical semigroups, and the latter in identifying shared cultural forms (such as "the friend of a friend is always an associate").

Linking algebraic and network homomorphisms. The factorisation of an algebra identifies relatively independent components of the algebra generated by constituents of the data array under the selected operation. It is natural to ask whether this analysis induces a corresponding decomposition of the network itself. That is, if an algebra A

has a homomorphic image B (such as one of the factors of A), is it possible to find one or more homomorphisms of the network that is consistent with the congruence relation σ for the homomorphism and that generates the algebra B ? If so, we can argue that there is an association between such a reduction of the network and the homomorphic image.

In the case of the lattice of an affiliation network, such a reduction can always be uniquely found [4]. For semigroups and role algebras, there is no guarantee that such a reduction can be found, but it is possible to identify the smallest homomorphic image of the network whose algebra contained B as a homomorphic image [36].

8. Examples

We illustrate a number of the algebraic constructions just described for a multiple network blockmodel and an approximation to an affiliation network.

Network semigroup. Table 1 corresponds to a multiple network on 7 nodes with two kinds of edges, F and N . The network is a blockmodel reported by Vickers for relations of friendship and negative ties among members of a high school class in an Australian country town [47]. The questions from which the blockmodel was constructed were (a) "Who are your best friends?" and (b) "Who would you rather not have as a friend?"

Table 1. A multiple network on 7 nodes

Relation	Block	Block						Relation	Block	Block								
		1	2	3	4	5	6			7	1	2	3	4	5	6	7	
F	1	1	1	1	0	0	0	N	1	0	0	0	1	1	1	1		
	2	0	1	0	0	0	0		1	2	0	0	0	1	1	1	0	
	3	1	1	1	0	0	0		0	3	0	0	0	1	1	1	0	
	4	0	0	0	1	0	0		0	4	0	1	1	0	0	0	1	1
	5	0	0	0	0	1	0		0	5	0	0	0	0	0	0	0	1
	6	1	0	0	0	0	1		0	6	0	0	0	1	0	0	0	1
	7	1	1	1	0	0	0		1	7	0	0	0	1	1	1	1	1

The semigroup generated by the multiple network in Table 1 has the right multiplication table and partial order shown in Table 2. The process of constructing the semigroup generates the sets $W_1 = \{F, N\}$, $W_2 = \{F^2, FN, NF, N^2\}$, $W_3 = \{F^3, FNF, FN^2, NF^2, N^2F, N^3\}$, $W_4 = \{FN^2F, NF^3\}$ and $W_5 = \emptyset$; equations include $F^2N = FN$, $NFN = FN^2$, and $F^4 = F^3$.

It is evident from Table 2 that the relations F and N both have index 3 and period 1. The relation F is regular, whereas the relation N is not.

Each of the equations of Table 2 specifies the empirical coincidence of potentially

distinct labelled walks. The equation $F^2N=FN$, for example, indicates the those individuals nominated by one's friends as preferred non-friends coincide with individuals so nominated by the friends of one's friends.

Table 2. Right multiplication table and partial order for the semigroup of the multiple network of Table 1

Right multiplication table				Partial order														
Element	Word	Generators		Elements														
		F	N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
1	F	3	4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	N	5	6	2	0	1	0	0	0	0	0	0	0	0	0	0	0	0
3	FF	7	4	3	1	0	1	0	0	0	0	0	0	0	0	0	0	0
4	FN	8	9	4	0	1	0	1	0	0	0	0	0	0	0	0	0	0
5	NF	10	9	5	0	1	0	0	1	0	0	0	0	0	0	0	0	0
6	NN	11	12	6	0	0	0	0	0	1	0	0	0	0	0	0	0	0
7	FFF	7	4	7	1	0	1	0	0	0	1	0	0	0	0	0	0	0
8	FNF	8	9	8	0	1	0	1	1	0	0	1	0	1	0	0	0	1
9	FNN	13	12	9	0	0	0	0	0	1	0	0	1	0	0	0	0	0
10	NFF	14	9	10	0	1	0	0	1	0	0	0	0	1	0	0	0	0
11	NNF	11	12	11	1	0	1	0	0	1	1	0	0	0	1	0	0	0
12	NNN	13	12	12	0	1	0	1	0	1	0	0	1	0	0	1	0	0
13	FNNF	13	12	13	1	1	1	1	1	1	1	1	1	1	1	1	1	1
14	NFFF	14	9	14	0	1	0	0	1	0	0	0	0	1	0	0	0	1

Role algebra. The role algebra for block 4 in the network of Table 1 is presented in Table 3. Equations in the right multiplication table indicate equalities among labelled paths having block 4 as the source. For example, the equation $FF = F$ indicates that friendship paths of length 2 emanating from block 4 reach exactly the same set of blocks as direct friendship ties. Likewise, the equation $FN = N$ allows us to infer that block 4 would prefer not to have as friends precisely those blocks so nominated by their friends. The partial order diagram indicates that block 4's friendship ties are a subset of those to whom block 4 has negative tie paths of length 2.

Table 3. The role algebra for block 4 in the network of Table 1

Right multiplication table				Partial order						
Element	word	Generators		elements						
		F	N	1	2	3	4	5	6	
1	F	1	2	1	1	0	0	0	0	0
2	N	3	4	2	0	1	0	0	0	0
3	NF	3	4	3	0	1	1	0	0	0
4	NN	5	6	4	1	0	0	1	0	0

5	NNF	5	6	5	1	1	1	1	1	1
6	NNN	5	6	6	1	1	0	1	0	1

The role algebra of block 4 has a unique factorisation. The lattice of π -relations has two atoms and each atom possesses a unique maximal meet-complement; the atoms and their maximal meet-complements are shown in Table 4. These unique maximal meet-complements are associated with the role algebras shown in Table 5.

In the first factor, both *FN* and *NF* ties from block 4 coincide with *N* ties, and *NNF* ties include all others. In the second factor, any path from block 4 whose last tie is labelled *N* coincides with an *N* tie.

Table 4. Atoms and unique maximal-meet complements for the block 4 role algebra

	Partial order Elements						Partial order Elements							
	Elements	1	2	3	4	5	6	Elements	1	2	3	4	5	6
Atom	1	1	0	0	0	0	0	1	1	0	0	0	0	0
	2	0	1	0	0	0	0	2	0	1	0	0	0	0
	3	0	1	1	0	0	0	3	0	1	1	0	0	0
	4	1	1	0	1	0	0	4	1	0	0	1	0	0
	5	1	1	1	1	1	1	5	1	1	1	1	1	1
	6	1	1	0	1	0	1	6	1	1	1	1	0	1
Maximal meet-complement	1	1	0	0	0	0	0	1	1	0	0	0	0	0
	2	0	1	1	0	0	0	2	1	1	0	1	0	1
	3	0	1	1	0	0	0	3	1	1	1	1	1	1
	4	1	0	0	1	0	0	4	1	1	0	1	0	1
	5	1	1	1	1	1	1	5	1	1	1	1	1	1
	6	1	1	1	1	1	1	6	1	1	0	1	0	1

Table 5. Factors of the role algebra for block 4

<i>Factor 1</i>		Right multiplication table		Partial order elements			
		Generators		elements			
Element	word	F	N	1	2	3	4
1	F	1	2	1	1	0	0
2	N	2	3	2	0	1	0
3	NN	4	4	3	1	0	1
4	NNF	4	4	4	1	1	1
<i>Factor 2</i>		Right multiplication table		Partial order elements			
		Generators		elements			
Element	word	F	N	1	2	3	

1	F	1	2	1	1	0	0
2	N	3	2	2	0	1	0
3	NF	3	2	3	0	1	1

Lattice of an affiliation network. The affiliation network in Table 6 is an approximation to the so-called “Southern Women” data [19, 27]. The approximation was obtained by a method described in [52], and is akin to a form of dual clustering of the rows and columns of the original data array. The Galois lattice of the (approximate) affiliation array is presented in Figure 1. In Figure 1, known as the *line diagram* of the lattice, one element s is drawn above and connected to another element t if s covers t , (that is if $s \geq t$ and there is no element u distinct from s and t for which $s \geq u \geq t$) [51]. The meet of two elements is the greatest element lying below both elements and connected to them by a descending path. The join of two elements is the least element to which they are both connected by an ascending path.

Table 6. An affiliation network: an approximation to the Southern Women data; the approximation differs from the original data in the 20 underlined values)

	event													
woman	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	<u>1</u>	1	1	0	0	0	0	0
2	1	1	1	<u>1</u>	1	1	1	1	<u>1</u>	0	0	0	0	0
3	<u>1</u>	1	1	1	1	1	1	1	1	0	0	0	0	0
4	<u>0</u>	0	1	1	1	1	1	1	0	0	0	0	0	0
5	<u>0</u>	0	1	1	1	<u>1</u>	1	<u>1</u>	0	0	0	0	0	0
6	0	0	1	0	1	1	0	1	0	0	0	0	0	0
7	0	0	0	0	<u>0</u>	<u>0</u>	1	1	0	0	0	0	0	0
8	0	0	0	0	<u>0</u>	<u>0</u>	0	1	1	0	0	0	0	0
9	0	0	0	0	<u>0</u>	0	1	1	1	0	0	0	0	0
10	0	0	0	0	0	0	1	1	1	0	0	<u>0</u>	0	0
11	0	0	0	0	0	0	0	1	<u>0</u>	1	0	1	0	0
12	0	0	0	0	0	0	0	1	1	1	0	1	1	1
13	0	0	0	0	0	0	<u>0</u>	1	1	1	0	1	1	1
14	0	0	0	0	0	<u>0</u>	1	<u>1</u>	1	1	1	1	1	1
15	0	0	0	0	0	0	1	1	0	1	1	1	0	0
16	0	0	0	0	0	0	0	1	1	0	0	0	0	0
17	0	0	0	0	0	0	0	<u>1</u>	1	0	<u>0</u>	0	0	0
18	0	0	0	0	0	0	0	<u>1</u>	1	0	<u>0</u>	0	0	0

The rows and columns of the affiliation array may be mapped to lattice elements in such a way that any row element (in this case, a woman) is at or above any column elements (in this case, events) with which it is affiliated; likewise, any column element (event) is at or below any affiliated row elements (women). It can be seen from Figure 1, for example, that all women attend event e8, whereas only women W1, W2 and W3 attend

events e1 and e2.

The lattice displayed in Figure 2 is a maximal homomorphic image of the lattice of Figure 1 and provides a simplification of the lattice. In this diagram, a division of the women in the network into three ordered clusters is apparent. One cluster comprises the women W1 to W6, with women W4, W5 and W6 attending only a subset of the events attended by women W1, W2 and W3. A second cluster comprises women W11 to W15, with women W11 and W15 attending only a subset of the events attended by women W12, W13 and W14. The third cluster of women comprises W7, W8, W16, W17 and W18 and is again divided into two sub-clusters, one of whom (W7) attends a subset of the events attended by the other (W8, W16, W17, W18). It is also clear from the lattice of figure 2 that the first two clusters of women attend a mixture of distinct events (e1 to e6 in the case of the first cluster, and e10 to e14 in the case of the second) as well as common events (e7 to e9), whereas the third cluster of women attend only those events also attended by women in the first two clusters (e7 to e9). This structural feature of the affiliation network is drawn out very clearly in the unique maximal homomorphic image of the lattice of Figure 2, shown in Figure 3.

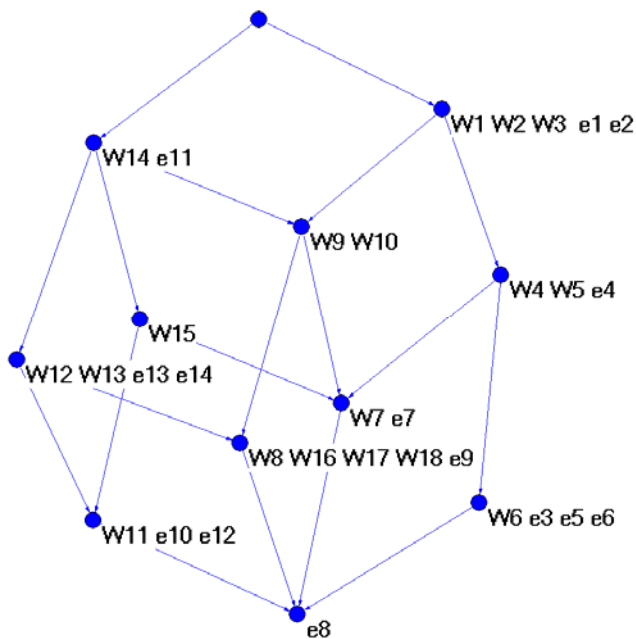


Figure 1. The line diagram of the Galois lattice of Table 4

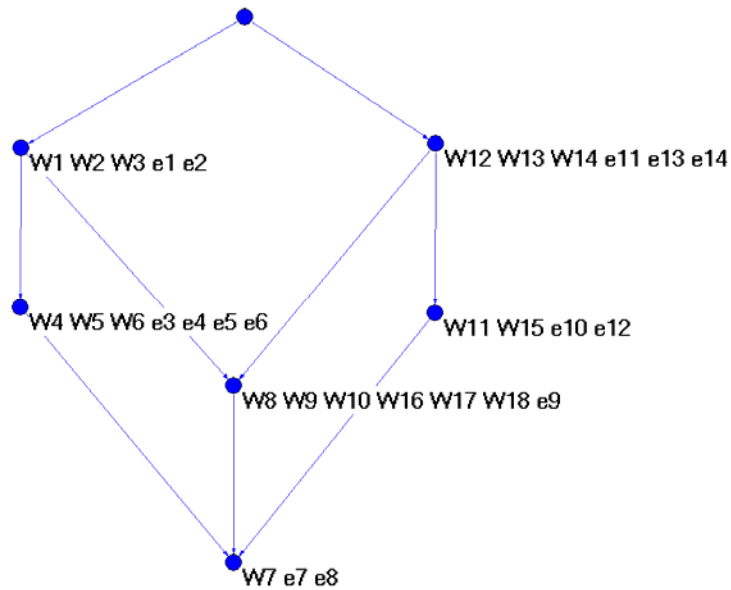


Figure 2. Largest homomorphic image of the lattice of Figure 1

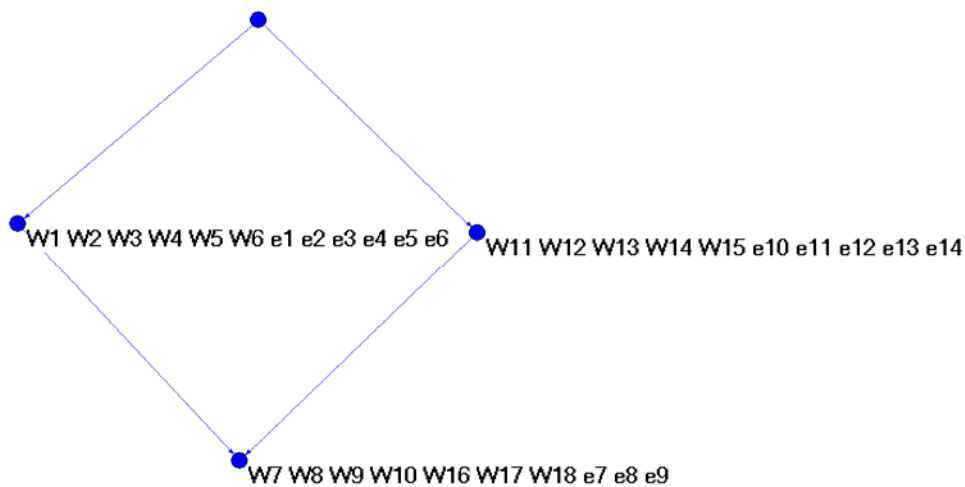


Figure 3. Largest homomorphic image of the lattice of Figure 2

9. Future directions

The algebraic constructions described above provide exact and detailed representations of structural relationships among network constituents. Of course, this level of detail assumes that the relational observations comprising the network are accurate, as no allowance is made for variability or error. This is often a tenuous assumption, and two important directions for further development utilise the structural

sensitivities of algebraic representations in the presence of network tie variability.

The first direction is to use statistical criteria to generate (partial) algebraic structures that summarise key structural features of a network [2,10,40,41]. Instead of the generalised Axiom of Quality introduced earlier, an *Approximate Axiom of Quality* is proposed in which $u \leq v$ in W whenever there is “sufficient evidence” that the relation $X_u \leq X_v$ holds. Such an approach can lead to a theoretically-guided and structurally-focussed form of exploratory data analysis for multiple networks. Theoretical guidance comes from choice of operations in the set F , while structural focus resides in the (partial) algebra to which the approach gives rise.

The second direction is to understand how exact algebraic representations can emerge as special, so-called *degenerate*, cases of statistical models such as exponential random graph models for networks (see the entry in this volume on *Exponential Random Graph Models for Networks*). An exponential random graph model defines a probability distribution on the set of all networks on a node set N as a function of some parameter vector. For any parameter vector, a subset of networks have what can be defined as minimum “energy”. As the parameter vector is scaled by an increasingly large multiplier, the probability associated with any network whose energy exceeds the minimum tends to zero. The minimum energy networks can therefore be seen as highly constrained or “frozen” structural forms associated with the stochastic model [5, 26, 42]. The structure of these forms will often warrant algebraic analysis of the type described earlier; in addition, we can understand the non-frozen models as stochastic generalisations of these structural forms.

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